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The α_s^3 correction to $\Gamma(Z^0 \to hadrons)$

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Abstract

We present the α_s^3 correction to the Z^0 decay rate into hadrons in the limit $m_{top} >> m_Z$.

1 Introduction

Precision measurements of the Z^0 decay rate into hadrons at LEP [1] provide precise means to extract the QCD coupling constant from experiment. The α_s^3 approximation to the Z^0 decay rate into hadrons is important for an accurate determination of α_s. The hadronic Z^0 decay rate is a sum of vector and axial-vector contributions of which the vector contribution is known to order α_s^3 from the calculation of $\sigma_{tot}(e^+e^- \to \gamma \to hadrons)$ [2]. The correctness of this calculation is strongly supported by [3] where the non-trivial connection between the result [2] and the α_s^3 approximation [4] to deep inelastic sum rules was established. The axial-vector part of the hadronic Z^0 decay rate was calculated to order α_s^2 in [5] and confirmed in [6] where renormalization group improvements were made. The Z^0 decay into 3 gluons in order α_s^3 has been calculated in [7]. In this paper we present the total α_s^3 correction to the Z^0 decay rate into hadrons by calculating the axial-vector part in order α_s^3 in the leading order of a large top mass expansion.

2 Preliminaries

For the Z^0 decay rate into hadrons, the quantity to be determined is the squared matrix element summed over all final hadronic states. One can express this quantity as the imaginary part of a current correlator in the standard way

$$\sum_h < 0|J^\mu|h >< h|J^\nu|0 > = 2Im\Pi^{\mu\nu}, \tag{1}$$

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$$\Pi^{\mu\nu} = i\int d^4z \mathrm{e}^{iq\cdot z} < 0|T(J^\nu(z)J^\mu(0))|0> = -g^{\mu\nu}\Pi_1(q^2) - q^\mu q^\nu \Pi_2(q^2). \tag{2}$$

Here $J^\mu = \frac{g}{4c_W}\sum_{q=1}^{6}\left[(2I_q^{(3)} - 4Q_q s_W^2)\overline{q}\gamma^\mu q + 2I_q^{(3)}\overline{q}\gamma^\mu\gamma_5 q\right]$ is the neutral weak quark current coupled to the Z^0 bozon in the Lagrangian of the Standard Model. The hadronic Z^0 decay width is expressed as

$$\Gamma_{had} \equiv \Gamma^V_{had} + \Gamma^A_{had} = \frac{1}{m_Z} Im\Pi_1(m_Z^2 + i0^+) \tag{3}$$

with the indicated decomposition into vector and axial-vector parts imposed by the structure of the neutral current.

Throughout this paper we use dimensional regularization [8] and the standard modification of the minimal subtraction scheme [9], the $\overline{MS}$ scheme [10]. For the treatment of the γ_5 matrix in dimensional regularization we use the technique described in [11] which is based on the original definition of γ_5 in [8]. We work in the approximation of 5 massless flavors and the top quark mass large compared to the Z^0 mass. We should stress that the top quark does not decouple [12] from the axial-vector part due to diagrams of the axial anomaly type.

The α_s^3 approximation [2] for the vector part in effective QCD with 5 active massless quark flavors in the $\overline{MS}$ scheme reads

$$\Gamma^V_{had} = \frac{G_F m_Z^3}{8\sqrt{2}\pi}\sum_{q=1}^{5}(2I_q^{(3)} - 4Q_q s_W^2)^2\left[1 + \frac{\alpha_s}{\pi} + 1.40923\left(\frac{\alpha_s}{\pi}\right)^2 - 12.76706\left(\frac{\alpha_s}{\pi}\right)^3\right] +$$
$$+ \frac{G_F m_Z^3}{8\sqrt{2}\pi}\left(\sum_{q=1}^{5}(2I_q^{(3)} - 4Q_q s_W^2)\right)^2\left[-.41318\left(\frac{\alpha_s}{\pi}\right)^3\right] \tag{4}$$

with the Fermi constant $G_F = \frac{g^2\sqrt{2}}{8c_W^2 m_Z^2}$. Here $\alpha_s = \alpha_s^{(5)}(m_Z)$ is the coupling constant in effective QCD with 5 active flavors (the expression for $\alpha_s^{(5)}$ in NNL approximation can be found e.g. in [2]).

It is convenient to split the axial-vector contribution in a non-singlet and a singlet part

$$\Gamma^A_{had} = \Gamma^{A,NS}_{had} + \Gamma^{A,S}_{had}. \tag{5}$$

The non-singlet part comes from Feynman diagrams where both axial vertices are located in one fermion loop. The non-singlet part can be reduced to the vector case by using the effective anticommutation property of the γ_5 matrix in the prescription that we use. The result in the effective theory with 5 active massless quark flavors is

$$\Gamma^{A,NS}_{had} = \frac{G_F m_Z^3}{8\sqrt{2}\pi}\sum_{q=1}^{5}(2I_q^{(3)})^2\left[1 + \frac{\alpha_s}{\pi} + 1.40923\left(\frac{\alpha_s}{\pi}\right)^2 - 12.76706\left(\frac{\alpha_s}{\pi}\right)^3\right]. \tag{6}$$

3 The calculation

The singlet contribution $\Gamma_{had}^{A,S}$ comes from diagrams where each axial vertex is located in a separate fermion loop. The 3-loop and 4-loop singlet diagrams that we need to calculate are drawn in figure 1.

Figure 1. The symbol $\otimes$ is used to indicate an axial vertex.

In the Standard Model quarks in a weak doublet couple with opposite sign to the Z^0 bozon in the axial part of the neutral current. That is why the contributions from light doublets add up to zero in the massless limit for singlet diagrams. The only non-zero contribution comes from the top-bottom doublet due to the large mass difference between top and bottom quarks. The massless diagrams (i.e. without top quark loops) were calculated with techniques similar to the ones that were used for the calculation of $\sigma_{tot}(e^+e^- \to \gamma \to hadrons)$ [2]. The R^*-operation [13] which subtracts both ultraviolet and infrared divergences was a necessary part of these techniques. For the calculation of massive diagrams (i.e. with one or more top quark loops) we applied an asymptotic expansion in a large top mass m_t. The theory of Euclidean asymptotic expansions was developed in [14, 15]. For the large mass expansion of the diagrams we use the techniques developed in [16, 17]. For the actual calculation we relied heavily on the symbolic manipulation program FORM [18]. Massless diagrams were calculated with the help of the package MINCER [19]. Massive diagrams were calculated with a specially designed package for massive vacuum

3-loop diagrams which uses algorithms from ref. [20]. Both packages are based on the 'integration by parts' method [21]. Details of the calculation will be given in a longer paper where we will also present the higher order terms in the large mass expansion. The computations were done in an arbitrary covariant gauge. The cancellation of the gauge dependence in physical quantities gives a good check of the results. The result of our calculation in the leading approximation of the large m_t expansion for 6 flavors is

$$\Gamma^{A,S}_{had} = \frac{G_F m_Z^3}{8\sqrt{2}\pi}\left[d_2\left(\frac{\alpha_s^{(6)}}{\pi}\right)^2 + d_3\left(\frac{\alpha_s^{(6)}}{\pi}\right)^3\right], \tag{7}$$

$$\begin{aligned}
d_2 &= T_F^2 D\left(-\tfrac{37}{24} + \tfrac{1}{2}ln(\tfrac{m_Z^2}{m_t^2})\right),\\
d_3 &= N_f T_F^3 D\left(\tfrac{25}{36} - \tfrac{1}{18}\pi^2 + \tfrac{1}{9}ln(\tfrac{\mu^2}{m_t^2}) - \tfrac{1}{6}ln^2(\tfrac{\mu^2}{m_t^2}) - \tfrac{11}{12}ln(\tfrac{m_Z^2}{\mu^2}) + \tfrac{1}{6}ln^2(\tfrac{m_Z^2}{\mu^2})\right)\\
&\quad + C_A T_F^2 D\left(-\tfrac{215}{48} - \tfrac{1}{2}\zeta_3 + \tfrac{11}{72}\pi^2 + \tfrac{19}{36}ln(\tfrac{\mu^2}{m_t^2}) + \tfrac{11}{24}ln^2(\tfrac{\mu^2}{m_t^2}) + \tfrac{161}{48}ln(\tfrac{m_Z^2}{\mu^2}) - \tfrac{11}{24}ln^2(\tfrac{m_Z^2}{\mu^2})\right)\\
&\quad + C_F T_F^2 D\left(-\tfrac{3}{4} + \tfrac{3}{2}\zeta_3 - \tfrac{3}{4}ln(\tfrac{\mu^2}{m_t^2})\right) + T_F^3 D\left(-\tfrac{157}{108} + \tfrac{1}{18}\pi^2 + \tfrac{11}{12}ln(\tfrac{m_Z^2}{m_t^2}) - \tfrac{1}{6}ln^2(\tfrac{m_Z^2}{m_t^2})\right).
\end{aligned}$$

Here $m_t \equiv m_t(\mu)$ is the running top mass in the $\overline{MS}$ scheme. $\alpha_s^{(6)}(\mu) = \frac{g^2}{4\pi}$ is the coupling constant in QCD with 6 flavors, $C_F = \frac{4}{3}$ and $C_A = 3$ are the Casimir operators of the fundamental and adjoint representation of the color group $SU(3)$, $D = 8$ is the dimension of the Lie algebra, $T_F = \frac{1}{2}$ is the trace normalization of the fundamental representation, $N_f = 6$ is the number of quark flavors, ζ is the Riemann zeta-function. Contributions with π^2 originate from terms containing $ln^3(-m_Z^2 - i0^+) = (ln(m_Z^2) - i\pi)^3$ when one takes the imaginary part of the correlator. Results for individual diagrams contain ζ_2, ζ_4 and ζ_5 but these contributions add up to zero in the total result. Note that the α_s^2 order agrees with the known calculation [5].

The coefficients of the logarithms are in agreement with the required renormalization group invariance of the physical quantity

$$\mu^2 \frac{d}{d\mu^2}\Gamma^{A,S}_{had}(\mu^2, \alpha_s(\mu), m_t(\mu)) = 0. \tag{8}$$

Of course the true physical quantity is Γ_{had}. But from a theoretical point of view the singlet part $\Gamma^{A,S}_{had}$ is renormalized independently of the non-singlet part and is therefore renormalization group invariant by itself.

The α_s^3 approximation for the vector part and the axial non-singlet part are calculated in effective QCD with 5 active massless flavors where the top quark decouples. Although the top quark does not decouple from the singlet axial part we should convert it to an expression in the effective theory to give a consistent total Γ_{had}. The QCD decoupling relations in two loop approximation in the $\overline{MS}$ scheme were calculated in [22, 23]. The

connection between the full coupling constant $\alpha_s^{(6)}$ and the coupling constant of effective QCD with 5 active flavors reads

$$\frac{\alpha_s^{(6)}(\mu)}{\pi} = \frac{\alpha_s^{(5)}(\mu)}{\pi} + \left(\frac{\alpha_s^{(5)}(\mu)}{\pi}\right)^2 \frac{T_F}{3} ln(\frac{\mu^2}{m_t^2}) + O(\alpha_s^3). \tag{9}$$

Substitution of this expression in (7) gives the expression for effective QCD

$$\Gamma_{had}^{A,S} = \frac{G_F m_Z^3}{8\sqrt{2}\pi} \left[\left(\frac{\alpha_s}{\pi}\right)^2 \left(-\frac{37}{12} + ln(\frac{m_Z^2}{m_t^2})\right) + \right.$$

$$\left. + \left(\frac{\alpha_s}{\pi}\right)^3 \left(-\frac{5651}{216} + \zeta_3 + \frac{23}{36}\pi^2 + \frac{31}{18} ln(\frac{\mu^2}{m_t^2}) + \frac{373}{24} ln(\frac{m_Z^2}{\mu^2}) + \frac{23}{12} ln^2(\frac{\mu^2}{m_t^2}) - \frac{23}{12} ln^2(\frac{m_Z^2}{\mu^2})\right)\right] \tag{10}$$

where $\alpha_s = \alpha_s^{(5)}(\mu)$ is the coupling constant in effective QCD with 5 flavors. The α_s^3 term drastically diminishes the μ-dependence of eq.(10) and makes it stable in a wide interval around $\mu \approx m_Z$.

Summing equations (4), (6) and (10) and putting the renormalization scale μ in the standard way equal to m_Z we get the hadronic Z^0 decay width (in the approximation of 5 massless quarks and a heavy top quark, in the leading order of the large m_t expansion)

$$\Gamma_{had} = \frac{G_F m_Z^3}{8\sqrt{2}\pi} \sum_{q=1}^{5} (2I_q^{(3)} - 4Q_q s_W^2)^2 \left[1 + \frac{\alpha_s}{\pi} + 1.40923 \left(\frac{\alpha_s}{\pi}\right)^2 - 12.76706 \left(\frac{\alpha_s}{\pi}\right)^3\right] +$$

$$+ \frac{G_F m_Z^3}{8\sqrt{2}\pi} \left(\sum_{q=1}^{5} (2I_q^{(3)} - 4Q_q s_W^2)\right)^2 \left[-.41318 \left(\frac{\alpha_s}{\pi}\right)^3\right] +$$

$$+ \frac{G_F m_Z^3}{8\sqrt{2}\pi} \sum_{q=1}^{5} (2I_q^{(3)})^2 \left[1 + \frac{\alpha_s}{\pi} + 1.40923 \left(\frac{\alpha_s}{\pi}\right)^2 - 12.76706 \left(\frac{\alpha_s}{\pi}\right)^3\right] +$$

$$+ \frac{G_F m_Z^3}{8\sqrt{2}\pi} \left[\left(\frac{\alpha_s}{\pi}\right)^2 \left(-\frac{37}{12} + ln(\frac{m_Z^2}{m_t^2})\right) + \left(\frac{\alpha_s}{\pi}\right)^3 \left(-18.65440 + \frac{31}{18} ln(\frac{m_Z^2}{m_t^2}) + \frac{23}{12} ln^2(\frac{m_Z^2}{m_t^2})\right)\right]. \tag{11}$$

Here and below $m_t \equiv m_t(m_Z)$ is the $\overline{MS}$ top mass at the scale m_Z. One may relate it to the pole mass through the expression $m_t(m_Z) = m_{pole}\left[1 - \frac{\alpha_s(m_Z)}{\pi}\left(ln(\frac{m_Z^2}{m_{pole}^2}) + \frac{4}{3}\right) + O(\alpha_s^2)\right]$ which is known in the NNL approximation [24] or relate it to $m_t(m_t)$ through the expression $m_t(m_Z) = m_t(m_t)\left[1 - \frac{\alpha_s(m_Z)}{\pi} ln(\frac{m_Z^2}{m_t^2(m_t)}) + O(\alpha_s^2)\right]$. This would correspondingly modify the coefficients of the α_s^3 term.

Substitution of the values of the physical parameters into eq.(11) from [25] gives (in the $\overline{MS}$ scheme)

$$\Gamma_{had}(GeV) = 1.671 \left[1 + \left(\frac{\alpha_s}{\pi}\right) + \left(\frac{\alpha_s}{\pi}\right)^2 \left(.9502 + .1489\, ln(\frac{m_Z^2}{m_t^2}) \right) + \right.$$
$$\left. + \left(\frac{\alpha_s}{\pi}\right)^3 \left(-15.650 + .2564\, ln(\frac{m_Z^2}{m_t^2}) + .2853\, ln^2(\frac{m_Z^2}{m_t^2}) \right) \right]. \qquad (12)$$

Note that the logarithms in the α_s^3 term tend to cancel each other for $m_t(m_Z) \approx 140 GeV$. For this value of m_t the calculated order α_s^3 singlet contribution to Γ_{had} adds about 20% to the previously known sum of vector and axial non-singlet contributions of order α_s^3.

From eq.(11) one can also obtain the α_s^3 approximation to $\sigma_{tot}(e^+e^- \to \gamma, Z^0 \to hadrons)$ in the energy range below the top quark threshold.

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